The Derrida–Retaux conjecture for recursive models

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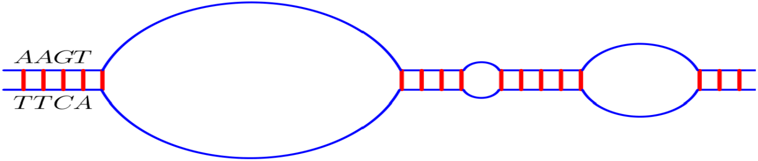
Outline of the talk

- A toy model: simpler to analyse
- Phase transition of Derrida-Retaux model
- Some conjectures and our results
- Main tool: Open paths of a binary tree

A toy model: simpler to analyse

Poland and Scheraga (1966): A model for the denaturation of the DNA molecule



Fix ω, then the partition function $Z_L(\omega)$ of a molecule is given by

$$Z_L(\omega) = \sum_{s \in \mathcal{S}_L,\ s_1 = s_L = 0} \exp\left(-\beta \sum_{i=1}^{L} \omega_i 1_{\{s_i = 0\}} \right),$$

where β is the inverse temperature and $\mathcal{S}_L$ is the set of possible pathes.

A toy model: simpler to analyse

Suppose $\omega_i, i \geq 1$ are i.i.d.. We will focus on the free energy

$$F(\beta) := \lim_{L\to\infty} \frac{\log(Z_L(\omega))}{L}.$$

The existence of the limit follows directly from the fact that for $L > N$,

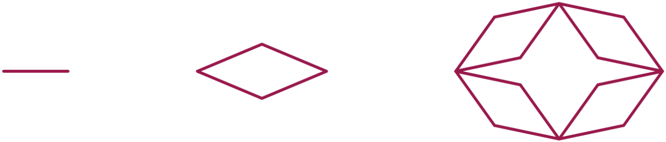
$$Z_L(\omega) \geq \sum_{s\in\mathcal{S}_L,\ s_1=s_L=0} 1_{\{s_N=0,\ s_{N+1}=0\}} \exp\left(-\beta \sum_{i=1}^{L} \omega_i 1_{\{s_i=0\}}\right)$$

$$= \sum_{s\in\mathcal{S}_L,\ s_1=s_N=0,\ s_{N+1}=s_L=0} \exp\left(-\beta \sum_{i=1}^{N} \omega_i 1_{\{s_i=0\}}\right) \exp\left(-\beta \sum_{i=1}^{L-N} \omega_{N+i} 1_{\{s_{N+i}=0\}}\right)$$

$$= Z_N(\omega) Z_{L-N}(\theta^N \omega).$$

A toy model: simpler to analyse

The problem will become simple if we consider it on diamond lattices.

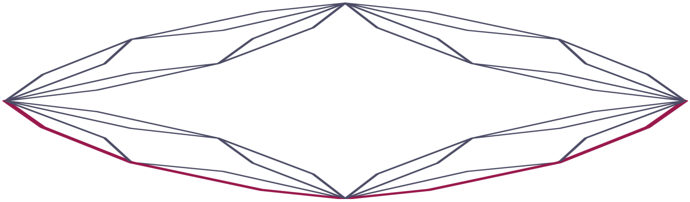


$n = 0$ $n = 1$ $n = 2$

Let L be the length of the n-th Diamond lattice, then $L = 2^n$.

Let B_L be the number of pathes with length L, then $B_L = 2^{L-1}$.

A toy model: simpler to analyse



Derrida-Hakim-Vannimenus (1992) gave the relation between the partition function Z_{2L} of a system of size $2L$ and its two subsystems by

$$Z_{2L} = Z_L^{(1)} Z_L^{(2)} + \frac{1}{2} B_{2L}.$$

Noting that $B_L = 2^{L-1}$ and $B_{2L} = 2^{2L-1}$, one has $\frac{Z_{2L}}{B_{2L}} = \frac{Z_L^{(1)} Z_L^{(2)}}{2B_L^2} + \frac{1}{2}$

Furthermore, we set $X_n = \frac{\log(Z_L/B_L)}{\log(2)}$ for $L = 2^n$, then

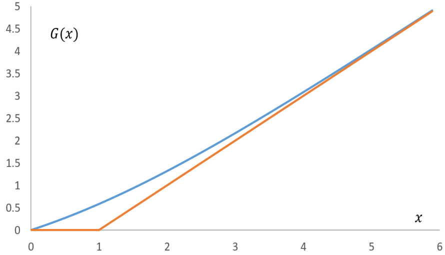
$$2^{X_{n+1}} = \frac{1}{2} 2^{X_n^{(1)} + X_n^{(2)}} + \frac{1}{2} \quad \Rightarrow \quad X_{n+1} = \mathcal{G}(X_n^{(1)} + X_n^{(2)}).$$

A toy model: simpler to analyse

So that we have a recursive system which satisfies

$$X_{n+1} = \mathcal{G}(X_n^{(1)} + X_n^{(2)}),\ n \geq 0,$$

where $X_n^{(1)}, X_n^{(1)}$ are independent copies of X_n and $\mathcal{G}(x) = x - 1 + \frac{\log(1+2^{-x})}{\log 2}$.



Derrida and Retaux (2014) simplify the model again,

$$X_{n+1} = \mathcal{G}(X_n^{(1)} + X_n^{(2)}) \approx \max\{X_n^{(1)} + X_n^{(2)} - 1,\ 0\}.$$

A toy model: simpler to analyse

Definition: Begin with a random variable $X_0 \geq 0$, and recursively

$$X_{n+1} = \max\{X_n^{(1)} + X_n^{(2)} - 1, 0\}, \quad n \geq 0,$$

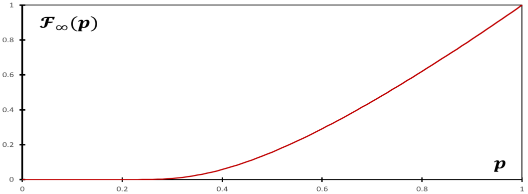
where $X_n^{(1)}, X_n^{(2)}$ are independent copies of X_n.

- Free energy: $\mathcal{F}_\infty := \lim_{n\to\infty} \downarrow \frac{\mathbf{E}(X_n)}{2^n} = \lim_{n\to\infty} \uparrow \frac{\mathbf{E}(X_n)-1}{2^n}$.

To study the phase transition, we parameterize the initial distribtuion:

$$\mathbf{P}(X_0 = 0) = 1 - p, \qquad \mathbf{P}(X_0 = 2) = p.$$

Phase transition of Derrida-Retaux model



Theorem (Collet-Eckmann-Glaser-Martin 1984): $\exists$ Critical value p_c,

$$\mathcal{F}_\infty(p) = 0, \quad p \le p_c \quad \text{and} \quad \mathcal{F}_\infty(p) > 0, \quad p > p_c.$$

For the initial distribution above, $p_c = \frac{1}{5}$!

Phase transition of Derrida-Retaux model

Part of Proof. Let $G_n(s) = \mathbf{E}_p(s^{X_n})$. By $X_{n+1} = (X_n^{(1)} + X_n^{(2)} - 1)^+$,

$$G_{n+1}(s) = \frac{1}{s}G_n(s)^2 + \frac{s-1}{s}G_n(0)^2.$$

- $G'_{n+1}(s) = \frac{2}{s}G'_n(s)G_n(s) - \frac{1}{s^2}G_n(s)^2 + \frac{1}{s^2}G_n(0)^2.$
- $2G'_{n+1}(2) - G_{n+1}(2) = G_n(2)[2G'_n(2) - G_n(2)].$

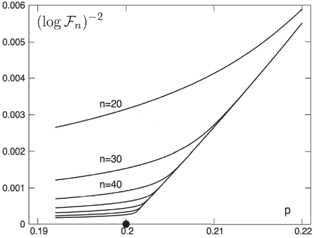
$$\text{So,} \quad 2G'_n(2) - G_n(2) \begin{cases} < 0, & \text{if } 2G'_0(2) - G_0(2) < 0; \\ = 0, & \text{if } 2G'_0(2) - G_0(2) = 0; \\ > 0, & \text{if } 2G'_0(2) - G_0(2) > 0. \end{cases}$$

Solving the equation $2G'_0(2) - G_0(2) = 0$ gives $p_c = \frac{1}{5}$.

Open Problem: What is the critical value p_c when $X_0 \notin \mathbb{Z}^+$?

Some conjectures and our results

$\mathcal{F}_n(p) = \frac{\mathbf{E}_p(X_n)}{2^n}$ and $(\log \mathcal{F}_n)^{-2} \to (\log \mathcal{F}_\infty)^{-2} \propto (p - p_c)$



Conjecture (Derrida-Retaux 2014) :

$$\mathcal{F}_\infty(p) = \exp\{-\frac{K + o(1)}{(p - p_c)^{1/2}}\}, \quad p \downarrow p_c.$$

- A transition of infinite order (Berezinskii-Kosterlitz-Thouless type)

Some conjectures

Conjecture (Collet et al. 1984, Derrida-Retaux 2014): As $n \to \infty$,

$$\mathbf{P}_{p_c}(X_n \geq 1) \sim \frac{4}{n^2}.$$

C-Derrida-Hu-Lifshits-Shi 2019:

$$\mathbf{P}_{p_c}(X_n = k | X_n \geq 1) \sim 2^{-k}$$

C-Dagard-Derrida-Shi 2020:

$$n^2 \cdot 2^k \mathbf{P}_{p_c}(X_n = k) \sim 4 \exp\{-\frac{2k}{n}\}$$

Some conjectures and our results

Theorem 1 (Chen-Dagard-Derrida-Hu-Lifshits-Shi, AOP 2021)

$$\mathcal{F}_\infty(p) = \exp\{-\frac{1}{(p-p_c)^{1/2+o(1)}}\},\ p \downarrow p_c.$$

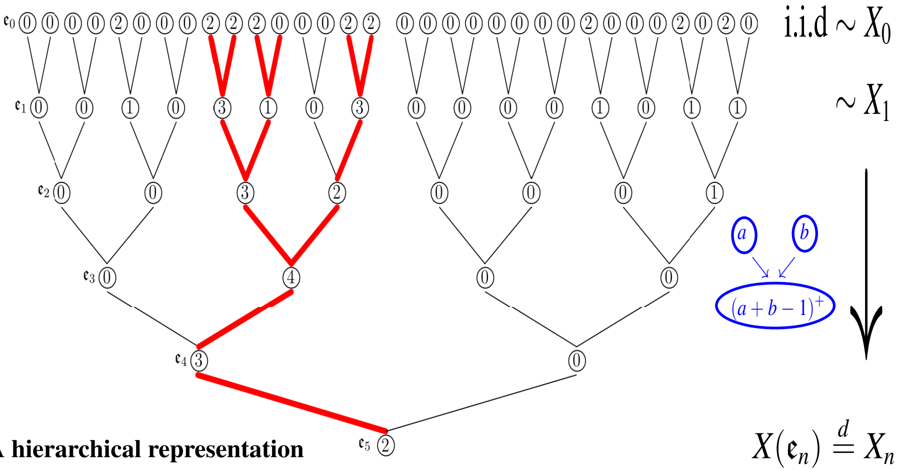
Theorem 2 (Chen-Hu-Shi, PTRF 2022): When $p = p_c$,

$$\mathbf{P}(X_n \neq 0) = \frac{1}{n^{2+o(1)}}, \quad \mathbf{E}(X_n) = \frac{1}{n^{2+o(1)}}, \quad n \to \infty.$$

Tool: A hierarchical representation of the Derrida-Retaux system

Main tool: Open paths of a binary tree

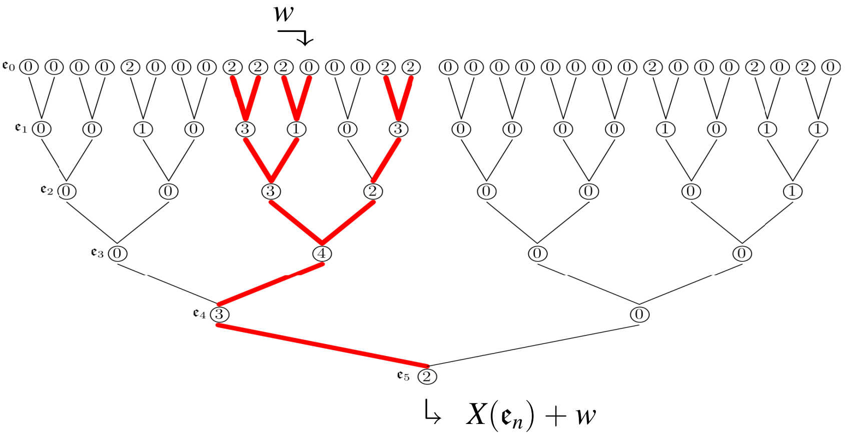
$$X(v) = \max\{X(v^{(1)}) + X(v^{(2)}) - 1,\ 0\}$$



A hierarchical representation

Main tool: Open paths of a binary tree

A path is open if along the path without taking the operation $x \to x^+$.



Let $N_n := \#$ open paths until the n-th generation.

Main tool: Open paths of a binary tree

Outline the proof of the theorem: $\mathscr{F}_\infty(p) = \exp\{-(p-p_c)^{-1/2+o(1)}\}$.

The main difficulty is to obtain $n_0 := \inf\{n : \mathbf{E}_p(X_n) > 1\}$, since

$$\mathcal{F}_\infty(p) \approx \frac{\mathbf{E}_p(X_{n_0})}{2^{n_0}} \approx e^{-\ln 2 \cdot n_0}.$$

Our idea is:

- In the nearly supercritical regime and n is not so large, to some sence

$$\mathbf{E}_p(X_n) \geq \mathbf{E}_{p_c}(X_n) + \mathbf{E}_{p_c}(N_n)(p-p_c) \approx N_n(p-p_c).$$

- At the critical regime, $N_n \approx n^2$ conditionally on $N_n \geq 1$.

So that we have $n_0 \leq (p-p_c)^{-1/2+o(1)}$ and obtain the lower bound.

Main tool: Open paths of a binary tree

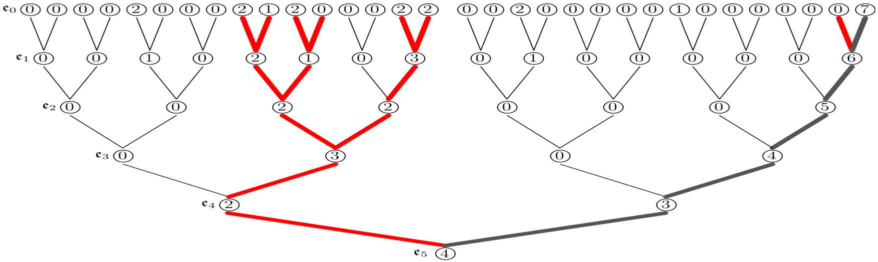
Other behaviours. As before, set $X_{n+1} = \max\{X_n^{(1)} + X_n^{(2)} - 1, 0\}$.

Fix $\alpha \in R$. Let $p \in (0,1)$. Assume that X_0 takes values in Z^+ and

$$\mathbf{P}(X_0 = 0) = 1 - p, \quad \mathbf{P}(X_0 \geq k) = pk^{-\alpha}2^{-k+1}, \quad k \geq 1.$$

Then

$$\mathbf{P}(\text{there exists } v \in \mathbb{T}_0^{\mathfrak{e}_n} \text{ which satisfies } X(v) \geq n) \geq pn^{-\alpha}.$$



Main tool: Open paths of a binary tree

Suppose now the system is at the critical regime.

- If $\alpha > 4$ then
$$N_n \approx n^2.$$
Indeed, a more general result holds when $\mathbf{E}(X_0^3 2^{X_0}) < \infty$.

- If $\alpha \in (2, 4]$, which is called the stable case, then
$$N_n \approx n^{\alpha - 2}.$$
See, Chen and Shi (2021).

- If $\alpha = 2$, then
$$p_c = 0.$$
See, Collet-Eckmann-Glaser-Martin (1984).

Main tool: Open paths of a binary tree

Theorem (Chen-Dagard-Derrida-Hu-Lifshits-Shi 2021): As $p \downarrow p_c$,

$$\mathcal{F}_\infty(p) = \begin{cases} \exp\{-(p-p_c)^{-\frac{1}{2}+o(1)}\}, & \alpha > 4; \\ \exp\{-(p-p_c)^{-\frac{1}{\alpha-2}+o(1)}\}, & \alpha \in (2,4]; \\ \exp\{-e^{-(C+o(1))/(p-p_c)}\}, & \alpha = 2. \end{cases}$$

Theorem (Hu and Shi 2018): If $\alpha < 2$ then

$$\mathcal{F}_\infty(p) = \exp\{-\frac{1}{(p-p_c)^{\frac{1}{2-\alpha}+o(1)}}\}, \quad p \downarrow p_c = 0.$$

Further Research

Conjecture: Derrida and Shi (2020)

Let $x > 0$ and $\{X_n\}$ be the critical Derrida-Retaux system with suitable integrable condition. Then conditionally on $\{X_n = [xn]\}$, $\frac{N_n}{n^2}$ converges in law as $n \to \infty$.

[1] Derrida and Shi (2020)

[2] Hu, Mallein and Pain (CMP 2020)

[3] Chen, Dagard, Derrida and Shi (JPA 2020)